Chapter 3.6.2

PICTURE (IM)PERFECT MATHEMATICS!

Wilfried Herget and Marlene Torres-Skoumal
Martin-Luther-University of Halle-Wittenberg, Germany, Email: herget@mathematik.uni-halle.de / Vienna International School, Austria, Email: mskoumal@vis.ac.at

Abstract: In this paper some unusual open-ended problems are presented, which have been "tried and tested" in secondary school. The main focus is not on calculation but rather on all the steps necessary before the calculations can begin. "Here is a situation. Think about it!" (Henry Pollak) Such exercises are indispensable toward the introduction of skills inherent in mathematical modelling where the emphasis is not on algorithmic procedures but rather on the higher order skills of translation, interpretation, and evaluation of the real life problem in terms of the mathematical model and its solution(s).

1. HERE IS A SITUATION – THINK ABOUT IT!

In the minds of the masses, "Doing Math means calculating". True, but this is certainly not the whole story. There is far more to mathematics than "mere" calculation! In this paper, some unusual open-ended problems will be presented which we successfully used in secondary school. In these tasks, calculating is not at the forefront, but rather all the thinking and planning skills necessary before the calculations can begin. "Here is a situation. Think about it!" (Henry Pollak) In the following exercise a newspaper article depicting a giant shoe is used as a starting point. "What size is this giant shoe?" Everyone seems to find a task like this rather unusual, and it is always intriguing to hear the different ways of solving the problem.

The person who fits this giant shoe must have enormous feet!

Antal Annus, a 73-year-old shoemaker from the Hungarian village of Csanádapáca, is depicted here, proudly presenting his hitherto most impressive “creation”. To this very day, we still do not know whether he really made the shoe for one of his customers.

Goslarsche Zeitung, 7.1.1995

Figure 3.6.2-1. What size is this giant shoe?

2. MANY DIFFERENT WAYS OF SOLVING THE PROBLEM

The standard approach is to use an object in the picture as an estimator or yardstick, e.g., the man’s glasses, his head, the width of the apron he is wearing, etc. It is quite easy to measure these things, both in the picture and in reality. A few simple calculations suffice to give us the length of the shoe. Once we have obtained the length of the shoe, however, we still do not know its real size! Have you ever thought about the relationship between the length of a shoe and the various parameters indicating a shoe's size? This could well turn out to be an interesting research project!

A colleague came up with another idea about how to solve the problem at hand. “A shoe is about the same length as a human face!” Assuming 42 to be the standard size shoe in Europe, we simply have to do our sums, providing the ratio *shoe length to shoe size* is linear. A school girl came up with another fascinating solution. Imagine that we turn the shoe at 90 degrees around the man's naval, we will then discover that the shoe is a little smaller than the man. If the man is about 1.70 m tall then the shoe must be approximately 1.5 m in length. Two students who conveyed their idea very clearly using body language put forth another possible solution: If we imagine the man in real life with his arms stretched out, he spans at least the length of the shoe. In the case of an average-sized human, this would be about 1.60 m. In reality, therefore, the shoe is approximately 1.5 m in length.

How reliable, however, are the different approaches to the problem? The results lie somewhere between 1 m and 2 m, so how accurate are in fact the various measurements and estimates? In the end, a critical comparison of each method might well reveal a slight difference but we still haven't come up with “the right solution”!

Finally, our task is to look at the relationship between normal shoe sizes and the length of the foot, in centimetres. Where do we start? One way would be to collect data by measuring various shoes. (Measuring big and small shoes lends itself well to homework since there are bound to be some “giants” and “dwarves” in the family and neighbourhood!) Another possibility, one which is rather unusual in math lessons, would be to make some enquiries in local shoe shops. If we are lucky, we might find some data on the shoe boxes themselves. Finally, we will get the type of relationship *normal shoe sizes ↔ length of the foot in centimetres* and some respective formula.

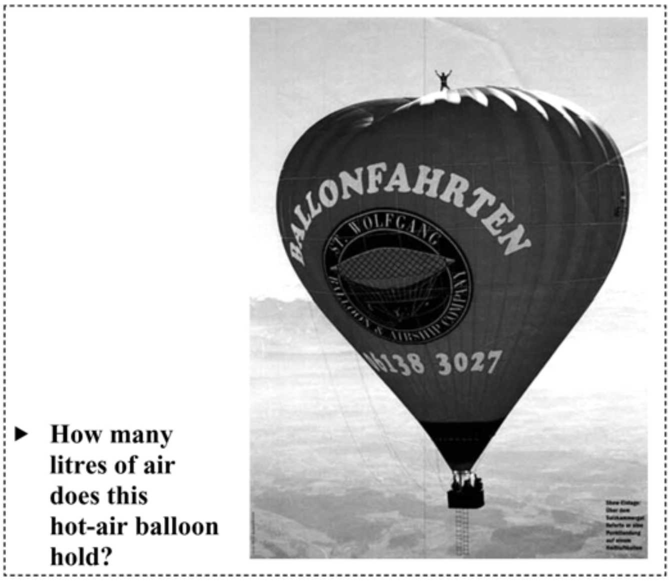
3. VERY PRECISE … AND VERY ROUGH!

Math lessons are typically characterised by precision. For example, if three sides of a rectangular box are 3 cm, 5 cm, and 7 cm respectively (and precisely, of course!), then find the volume of the box. But this obsession becomes an exercise in futility the moment mathematics becomes involved with “the rest of the world”. There, most of the numbers which crop up are only approximately correct. This is inevitable and unavoidable! Likewise, the results are only rough estimates.

In our lessons, therefore, one of our tasks, indeed obligations, should be to bridge the gap between these two different worlds: the world of accuracy so typical of mathematics, and that of lack of precision in the rest of the world. This is imperative because both worlds are important and both are indispensable. How can we possibly learn the true value of the precision and certainty of mathematics if we have not yet learnt that, in the “rest of the world”, this precision and reliability is something which is very difficult to achieve? On the other hand, one can only learn to cope well with this inaccuracy and blatant lack of precision if one has learned to exploit the many possibilities offered by the very precise field of mathematics.

4. A PICTURE TELLS A STORY OF WELL OVER 1,000 WORDS!

How is it then possible to bridge the gap between mathematics and the "rest of the world"? How do we carefully and sensitively introduce the young student to the uncertain world of mathematical modelling? At this point we propose a very special method: setting tasks mostly based on rather unusual newspaper cuttings that we are apt to call "Pictorial Problems" or "Picture Mathematics". Many tasks based on real-life situations are often far too cluttered with text to be truly effective for the young mathematics student. This is where a picture, supplemented by the students' general knowledge and imagination, comes in handy: "A picture can indeed say far more than a thousand words!"



Stern, 44/2001

Figure 3.6.2-2. The hot-air balloon

How many litres of air does this hot-air balloon hold?

At first, of course, to solve this task a model of the hot-air balloon should be made as accurately as possible, with the help of an object that is easy to describe. The wider the range of mathematical instruments available, the more instruments can be used to solve this task.

In a Calculus course, the interpretation of the set task would be based on rotated solids. Modern pocket calculators, with built-in programmes for regression analysis, have no limits whatsoever with regard to the type of function used. Indeed students left to their own resources showed untiring efforts in their quest for the "best" approximation. At the tender age of 15, 9^{th} graders were researching how to calculate volumes of revolution using a computer algebra system, although not fully understanding, of course, why such a procedure of integrating the squares of functions between two parameters and multiplying by π would be an effective method of approximating a volume since this topic formally comes later in their mathematical experience.

Other students were satisfied with much simpler solutions. They simply looked at familiar geometrical shapes in their model building kit and selected something suitable.

For example, one model of the balloon is made of the upper section using the shape of a hemisphere and the lower part is then made using a cylindrical cone. Another model consists of "chiselling" the balloon into a hemisphere, a frustrum of a cone for the middle section, and then a cone. In both cases the people in the photo are used as yardsticks. Other solutions may use even simpler models of the balloon: Let us take a big sphere as a suitable substitute, or even a cube – and it works, indeed!

When taking measurements we are obviously very much aware of how inaccurate these values are. There is little point in making note of the figures which appear after the decimal point as shown on the pocket calculator. Some calculations using upper and lower values should be made arriving at an "interval of tolerance" as an answer. Finally we obtain for the total volume of the balloon roughly 7,000 cubic meters.

5. DIFFERENT WAYS BUT COMMON IDEAS

Let us now itemize the steps inherent in the process of seeking solutions to these examples:

- "Real world" mathematics remains the focal point for the duration of the activity until a solution is reached – the problems do not exist merely as a desperate attempt to superimpose a real world problem on analytical techniques previously learned.
- The facts are analyzed and the mathematically relevant details are filtered out while the perhaps interesting, but irrelevant information (for the solution's sake) is laid aside.
- An appropriate object is chosen to serve as a yardstick for the necessary measurements which have to be made in the solution process.
- Necessary simplifications are performed.
- The interesting measurements are taken from the picture; through the measurement process one is constantly conscious of the unavoidable element of uncertainty yielded through the approximations.

- Common knowledge is activated, e.g., how tall is an average person, do body parts exist in certain proportions, etc. If necessary, information from other sources will be obtained.
- The relationship between the chosen yardstick and the measurements obtained will be mathematically defined and refined.
- A suitable mathematical model and methodology for the solution of the problem will emerge from this process, as opposed to students being handed pre-conceived ones. The students may choose the model they feel is most suitable, i.e., they must choose the model themselves.
- Technology allows for solutions previously denied students until much later in their mathematical development, or not at all.
- This entire process is guided and enlightened by fundamental mathematical considerations, strategies and concepts, which make a solution possible.
- Throughout this process other questions or ideas emerge, mathematical or otherwise, which can then be expanded upon, time permitting.

A discussion of these examples highlights the essential aspects of the process of mathematical modelling. In mathematics education knowledge and skills are necessary pre-requisites which assist us in various stages of the process, but to accomplish the entire task at hand, certain central ideas or concepts are necessary, namely the concepts of measurement, approximation, and linearization.

All of the above is in accordance with Freudenthal's view of mathematics (Freudenthal, 1968) – 'mathematizing' as the activity of looking for problems and solving them, by organizing all the information you have about this problem situation and then choosing and using suitable mathematical tools.

When using these "picture mathematics" exercises, the role of the teacher in the classroom changes from being mainly the disseminator of information to becoming a moderator or facilitator of knowledge. The teacher must carefully consider the various methodologies chosen by the students, and gently guide and direct their efforts in their quest for a solution. (For example, the most common error for wide discrepancies of approximations was incorrect handling of units, e.g., incorrectly changing cm^3 into m^3, or m^3 into liters, etc.) Furthermore, the teacher should encourage discussion and reflection on the various strategies employed, and point out the central concepts and ideas contained in the various solutions. Now more than ever expertise is needed in handling information overflow, performing thorough research, discerning the important from the unimportant and the correct from the questionable. Good communication skills are required in order to make the procedures and processes accessible to others. Lastly, all of the incorporated and integrated information should lead to a higher level of knowledge.

6. MODELLING AS A CENTRAL THEME

Nowadays, tasks requiring pure technical calculation can be solved with the help of a calculator or computer software. Consequently, more demanding activities are gaining importance, e.g., the analysis of problems affecting the "real world", i.e., mathematical modelling. It is to the child's benefit to discover the "discomforts" of uncertainty in mathematics as early as possible. For students as well as for teachers the shift from problem solving (one correct answer) to mathematical modelling (multi-solution paths leading to approximate answers each with possible limitations or potential for extension) requires a new set of teaching and learning skills – symbolizing data, cleverly translating the task into the language of mathematics, i.e., into a mathematical model, the internal treatment of this problem in the field of mathematics right up to its solution(s) possibly with the aid of technology, and finally, a deliberate interpretation and critical examination of the results obtained. "Has our original question really been answered? How accurate and how reliable is the result? How applicable are my results to other (new) situations?" Thus using carefully selected examples, the typical process of mathematical modelling may become a central theme in the classroom, including both modelling methods and the accuracy of the mathematics involved, of course without losing sight of the discrepancy between the mathematical model and reality.

7. ASSESSMENT OF MODELLING TASKS

We all know that in good classroom practice assessment is a natural by-product of the classroom experience and should be a celebration of achievement. In using such activities for assessment purposes we recommend, therefore, a criteria-based assessment that is defined from the sub-tasks themselves. Since in these open-ended tasks "many roads lead to Rome", the focus of the assessment should be uppermost on the selection of the path and the markers that the student has set up along the way to ensure a secure journey toward the goal. Considerations as criteria should therefore be – *Communication:* To what degree has the student used tables, diagrams, graphs, etc. as aids in defining the problem? Has the student used correct mathematical notation and terminology throughout the activity? (*Yes*, this is still very important, perhaps now more than ever, the need for correct and clear communication.) *The Model:* What has the student used as a yardstick, and is it well justified? Has the student taken into account all vital variables to the problem? Does the model suit the problem – how *well* does it fit the problem? *Mathematical Content:* Are the calculations correct and justified? Were

formulas correctly applied? *Evaluation:* To what extent has the student evaluated the meaningfulness of the approximations obtained from the model in light of the real life problem? Has the student considered limitations of the model, or possible extensions and applications of it?
The assessment of such activities can provide students with opportunities and rewards for carrying out mathematics without time limitations of tests and exams and their accompanying stresses. Furthermore it can provide the student who has difficulty performing well on traditional exams a sense of success and achievement in this subject. Hence, the emphasis in this kind of assessment should be on good mathematical writing and thoughtful reflection.

8. CARRY ON COLLECTING AND COMPILING

Additional information and many further examples are available under Herget (2000, 2002), Herget, Jahnke, & Kroll (2001) and Herget & Klika (2003). But, dear colleagues, you will surely find up-to-date pictures, perhaps even in your local newspaper, featuring events which are of interest to the pupils, and are closely related to *their* world.

REFERENCES

Freudenthal, H. (1968). Why to teach mathematics so as to be useful? *Educational Studies in Mathematics 1*, 3-8.

Herget, W. (2000). Wie groß? Wie hoch? Wie schwer? Wie viele? – Mathe-Welt. *mathematik lehren 101*, 23-46.

Herget, W. (2002). “Pictorial Problems”. One Question, but Many Ways, and Many Different Answers. In Weigand, H.-G., Neill, N., Peter-Koop, A., Reiss, K., Törner, G., & Wollring, B. (Eds.), *Developments in Mathematics Education in German-speaking Countries. Selected Papers from the Annual Conference on Didactics of Mathematics, Potsdam 2000*, (pp. 76-87). Hildesheim, Berlin: Franzbecker.

Herget, W., Jahnke, T., & Kroll, W. (2001). *Produktive Aufgaben für den Mathematikunterricht in der Sekundarstufe I.* Berlin: Cornelsen.

Herget, W., & Klika, M. (2003). Fotos und Fragen. Messen, Schätzen, Überlegen – viele Wege, viele Ideen, viele Antworten. *mathematik lehren 119*, 14-19.

International Baccalaureate Organization (1998). Internal Assessment: the Portfolio. In *Teachers Support Material: Mathematics Higher Level,* (pp. 47-58).